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# Quantum phases of a qutrit

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## Abstract

We consider various approaches to treat the phases of a qutrit. Although it is possible to represent qutrits in a convenient geometrical manner by resorting to a generalization of the Poincaré sphere, we argue that the appropriate way of dealing with this problem is through phase operators associated with the algebra  $su(3)$ . The rather unusual properties of these phases are caused by the small dimension of the system and are explored in detail. We also examine the positive operator-valued measures that can describe the qutrit phase properties.

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## 1. Introduction

The emerging field of quantum information, which embraces areas of futuristic technology such as quantum computing, quantum cryptography and quantum communications, has been built on the concepts of entanglement and qubits [1, 2]. The full appreciation of the complex quantal properties of these two ideas has provided powerful physical resources for new schemes that herald results that cannot be achieved classically [3].

Recently, the exploration of higher dimensional quantum systems has finally received the attention it rightly deserves. One could think that this represents a mere digression in a hot topic. However, qutrits have several interesting properties worth exploring [4]: the efficiency and security of many quantum information protocols are improved using qutrits [5–7], and larger violations of nonlocality via Bell tests are expected to occur for systems of entangled qutrits [8, 9].

In the modern parlance of quantum information the concept of phase for a qubit (or a qutrit) is ubiquitous. However, this notion is rather imprecise. Phases for three-level systems have been handled by invoking fuzzy concepts such as the phase of the associated

wavefunction [10]. Sometimes, the problem is reduced to the optimal estimation of the value of the phase shift undergone by the qutrit [11].

When comparing phases of two states, it is usually assumed that the relative phase is obtained from the argument of their inner product. In this perspective, the phase is considered as a state parameter. In recent years, we have learned that this relative phase shift can be of various origins, namely, it can be purely dynamical or purely geometrical or both. Presently, there is an immense interest in geometric phases in quantum optics [12, 13], especially in connection with quantum computing applications [14]. In fact, these phases are linked to the geometry of the state space: for a qubit, this space is the coset space  $SU(2)/U(1)$ , the well-known Poincaré sphere, while for a qutrit, a geometrical picture of the corresponding generalization to  $SU(3)/U(2)$  has been recently presented [15].

We emphasize that these notions, though well established in the classical limit, are not easily extrapolated into the realm of the quantum world. Since the phase is a physical property, it must, in the orthodox picture of quantum mechanics, be associated with a selfadjoint operator or at least with a family of positive operator-valued measures (POVMs). In this spirit, phase operators for the algebra  $su(2)$ , which describes qubits, have been previously worked out [16–19], as well as the optimal POVM for this problem [20]. The main goal of this paper is to work out a nontrivial extension to  $su(3)$  of the results available for  $su(2)$ , enabling us to introduce phase operators for qutrits with a clear physical picture. This seems of such fundamental importance that it is surprising that such a task was not undertaken a long time ago. We thus trust that this will be of relevance to workers in the various experimental fields currently under consideration for quantum computing technology and in quantum optics, in general.

## 2. Poincaré sphere for a qutrit

We first briefly recall the salient features of the Poincaré sphere representation for a qubit, with a view to preparing its generalization for a qutrit along the lines of [15]. A qubit lives in a two-dimensional complex Hilbert space  $\mathcal{H}^{(2)}$  spanned by two states:  $|1\rangle$  and  $|2\rangle$ . To get a useful parametrization of the state space of a general qubit described by the density matrix  $\hat{\rho}$ , we observe that the Pauli matrices  $\hat{\sigma}_a$  together with the identity  $\hat{1}$  form a complete set of linearly independent observables, and that any selfadjoint (trace class) operator can then be written as

$$\hat{\rho} = \frac{1}{2}(\hat{1} + \mathbf{n} \cdot \hat{\sigma}). \quad (1)$$

The physical condition  $\hat{\rho} \geq 0$  holds only when  $|\mathbf{n}| \leq 1$ . Hence, the state space coincides with the Bloch ball, and the set of pure states ( $\hat{\rho}^2 = \hat{\rho}$ ) with the boundary of this ball  $|\mathbf{n}| = 1$ , which is the Bloch sphere  $S^2$ . The general pure state

$$|\Psi\rangle = \sin(\theta/2)|1\rangle + e^{i\phi} \cos(\theta/2)|2\rangle \quad (2)$$

is represented by the unit vector

$$\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \quad (3)$$

The angle  $\theta$  is obviously related to the proportion of  $|1\rangle$  and  $|2\rangle$  in the composition of the state, while the parameter  $\phi$  is routinely interpreted as the quantum phase associated with the qubit and canonically conjugate to the inversion  $\hat{\sigma}_z$  [21]. We note in passing that diametrically opposite points on  $S^2$  correspond to mutually orthogonal vectors in  $\mathcal{H}^{(2)}$ .

Consider now a qutrit, living in a three-dimensional complex Hilbert space  $\mathcal{H}^{(3)}$  spanned by  $|1\rangle$ ,  $|2\rangle$  and  $|3\rangle$ . The roles of  $SU(2)$  and the Pauli matrices are now played by the group

$SU(3)$  and the eight generators of the corresponding  $su(3)$  algebra. A convenient set of Hermitian generators are the Gell-Mann matrices [22, 23]  $\hat{\lambda}_r$  ( $r = 1, \dots, 8$ ), which obey the commutation relations

$$[\hat{\lambda}_r, \hat{\lambda}_s] = 2i f_{rst} \hat{\lambda}_t \tag{4}$$

where, above and in the following, the summation over repeated indices applies. The structure constants  $f_{rst}$  are elements of a completely antisymmetric tensor spelled out explicitly in [15], for example.

A particular feature of the generators of  $SU(3)$  in the defining  $3 \times 3$  matrix representation is closure under anticommutation [23]

$$\{\hat{\lambda}_r, \hat{\lambda}_s\} = \frac{4}{3} \delta_{rs} \hat{\mathbb{1}} + 2d_{rst} \hat{\lambda}_t \tag{5}$$

where now  $d_{rst}$  form a totally symmetric tensor.

For the following, a vector-type notation is useful, based on the structure constants. The  $f$  and  $d$  symbols allow us to define both antisymmetric and symmetric products by

$$\begin{aligned} (\mathbf{A} \wedge \mathbf{B})_r &= f_{rst} A_s B_t = -(\mathbf{B} \wedge \mathbf{A})_r \\ (\mathbf{A} \star \mathbf{B})_r &= \sqrt{3} d_{rst} A_s B_t = +(\mathbf{B} \star \mathbf{A})_r. \end{aligned} \tag{6}$$

Given a density matrix  $\hat{\rho}$  we can expand it in terms of the unit matrix  $\hat{\mathbb{1}}$  and the  $\hat{\lambda}_r$  in the form

$$\hat{\rho} = \frac{1}{3} (1 + \sqrt{3} \mathbf{n} \cdot \hat{\boldsymbol{\lambda}}). \tag{7}$$

This is the equivalent to the Bloch ball for a qutrit. For a pure state the analogous Bloch sphere is defined by the condition

$$\mathbf{n} \cdot \mathbf{n} = 1, \quad \mathbf{n} \star \mathbf{n} = \mathbf{n}. \tag{8}$$

Thus, each pure qutrit state corresponds to a unique unit vector  $\mathbf{n} \in S^7$ , defining the seven-dimensional unit sphere. In addition, this vector must obey the condition  $\mathbf{n} \star \mathbf{n} = \mathbf{n}$ , which places three additional constraints, thus reducing the number of real parameters required to specify a pure state from the seven parameters needed to specify an arbitrary eight-dimensional vector to four.

In view of our discussion for qubits, it is clear that normalization and a choice for the arbitrary overall phase allow us to write these four parameters for any pure state as

$$|\Psi\rangle = \sin(\xi/2) \cos(\theta/2) |1\rangle + e^{i\phi_{12}} \sin(\xi/2) \sin(\theta/2) |2\rangle + e^{i\phi_{13}} \cos(\xi/2) |3\rangle. \tag{9}$$

Again,  $\theta$  and  $\xi$  determine the magnitudes of the components of  $|\Psi\rangle$ , while we can interpret  $\phi_{12}$  as the phase of  $|1\rangle$  relative to  $|2\rangle$  and analogously for  $\phi_{13}$ . We can easily obtain the expressions for  $\mathbf{n}$  in these local coordinates.

Some interesting geometric properties of this Poincaré sphere are discussed in [4]. In particular, it is easily seen that for two unit vectors  $\mathbf{n}$  and  $\mathbf{n}'$  representing pure states

$$0 \leq \arccos(\mathbf{n} \cdot \mathbf{n}') \leq \frac{2\pi}{3} \tag{10}$$

so mutually orthogonal vectors in  $\mathcal{H}^{(3)}$  do not lead to antipodal or diametrically opposite points on the Poincaré sphere, but to points with a maximum opening angle of  $2\pi/3$ .

### 3. Phase operators for a qutrit

Although there is a widespread usage of dealing with the qutrit phases as state parameters, this is not an orthodox way of proceeding, according to the very basic principles of quantum mechanics.

To gain further insights into this obvious although almost unnoticed point, we stress that the complete description of a qutrit involves the nine operators

$$\hat{S}_{ij} = |i\rangle\langle j| \quad (11)$$

where  $|i\rangle$  is a basis vector in  $\mathcal{H}^{(3)}$ . The three ‘diagonal’ operators  $\hat{S}_{ii}$  measure level populations, while the ‘off-diagonal’ ones  $\hat{S}_{ij}$  represent transitions from  $j$  to level  $i$ . One can easily check that they satisfy

$$[\hat{S}_{ij}, \hat{S}_{kl}] = \delta_{jk}\hat{S}_{il} - \delta_{il}\hat{S}_{kj} \quad (12)$$

which are the commutation relations of the algebra  $u(3)$  [24].

Because of the trivial constraint  $\hat{S}_{11} + \hat{S}_{22} + \hat{S}_{33} = \hat{\mathbb{1}}$ , only two populations can vary independently. For this reason, we shall work with two independent traceless operators

$$\hat{S}_{12}^z = \frac{1}{2}(\hat{S}_{22} - \hat{S}_{11}) \quad \hat{S}_{23}^z = \frac{1}{2}(\hat{S}_{33} - \hat{S}_{22}) \quad (13)$$

that measure atomic inversions between the corresponding levels. In atomic systems, the selection rules usually rule out one of the transitions and therefore the two independent inversions are automatically fixed. For a general qutrit, these inversions can be arbitrarily chosen.

The commuting operators  $\hat{S}_{12}^z$  and  $\hat{S}_{23}^z$  constitute a maximal Abelian subalgebra for the qutrit (known as Cartan subalgebra). From the discussion of the previous section, we expect  $\hat{S}_{12}^z$  and  $\hat{S}_{23}^z$  to be conjugate to the corresponding (independent) phases of the qutrit.

Note that  $(\hat{S}_{12}, \hat{S}_{12}^z)$  and  $(\hat{S}_{23}, \hat{S}_{23}^z)$  correspond to the qubits  $1 \leftrightarrow 2$  and  $2 \leftrightarrow 3$ . However, these two qubits are not independent, since equation (12) imposes highly nontrivial coupling between them.

At the operator level, the equivalent to the decomposition of a complex number in terms of modulus and phase is a polar decomposition [25]. Since  $\hat{S}_{21} = \hat{S}_{12}^\dagger$ , it seems appropriate to define [26]

$$\hat{S}_{12} = \hat{R}_{12} \hat{E}_{12} \quad (14)$$

where the ‘modulus’ is  $\hat{R}_{12} = \sqrt{\hat{S}_{12}\hat{S}_{21}}$  and  $\hat{E}_{12} = \exp(i\hat{\phi}_{12})$ ,  $\hat{\phi}_{12}$  being the Hermitian operator representing the phase.

One can easily work out that a unitary solution of equation (14) is given, up to an overall phase, by

$$\hat{E}_{12} = |1\rangle\langle 2| + e^{i\phi_0}|2\rangle\langle 1| - e^{-i\phi_0}|3\rangle\langle 3| \quad (15)$$

where the undefined factor  $e^{i\phi_0}$  appears due to the unitarity requirement of  $\hat{E}_{12}$ . The main features of this operator are largely independent of  $\phi_0$ , but for the sake of concreteness, we can make a definite choice. For example [19], for a qubit defined by a linear superposition of the states  $|1\rangle$  and  $|2\rangle$ , the complex conjugation of the wavefunction should reverse the sign of  $\hat{\phi}_{12}$ , which immediately leads to the condition  $e^{i\phi_0} = -1$ . We conclude then that a unitary phase operator that preserves the polar decomposition of equation (14) can be represented as

$$\hat{E}_{12} = |1\rangle\langle 2| - |2\rangle\langle 1| + |3\rangle\langle 3|. \quad (16)$$

The eigenstates of  $\hat{\phi}_{12}$  are those of  $\hat{E}_{12}$ , and easily found to be

$$|\phi_{12}^0\rangle = |3\rangle \quad |\phi_{12}^\pm\rangle = \frac{1}{\sqrt{2}}(|2\rangle \pm i|1\rangle) \quad (17)$$

with the corresponding eigenvalues of  $\hat{\phi}_{12}$ , 0 and  $\mp\pi/2$ , respectively. This is a remarkable result. It shows that the eigenvectors  $|\phi_{12}^{\pm}\rangle$  look like the standard ones for a qubit. However, the ‘spectator’ level  $|3\rangle$  is an eigenstate of this operator, which introduces drastic changes. In other words, the phase of the qubit  $1 \leftrightarrow 2$  ‘feels’ the state  $|3\rangle$ .

An analogous reasoning for the transition  $2 \leftrightarrow 3$  gives the corresponding operator  $\hat{E}_{23}$

$$\hat{E}_{23} = |2\rangle\langle 3| - |3\rangle\langle 2| + |1\rangle\langle 1| \quad (18)$$

with eigenvectors

$$|\phi_{23}^0\rangle = |1\rangle \quad |\phi_{23}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|3\rangle \pm i|2\rangle) \quad (19)$$

and the same spectrum as before.

As for the operator  $E_{13}$ , one must be careful, because it connects the lowest to the highest vector. In fact, the polar decomposition in this case gives as a unitary solution

$$\hat{E}_{13} = a|3\rangle\langle 2| - b^*|3\rangle\langle 1| + b|2\rangle\langle 2| + a^*|2\rangle\langle 1| + |1\rangle\langle 3| \quad (20)$$

with the condition  $|a|^2 + |b|^2 = 1$ . There are also nonunitary solutions to the polar decomposition, but they lack the interest to describe a phase observable in our context.

Note that the general solution (20) has the desirable property  $\hat{E}_{13}|3\rangle = |1\rangle$ . On physical grounds, we argue that the state  $|2\rangle$  should be a ‘spectator’ for the transition  $1 \leftrightarrow 3$ . Thus we impose  $\hat{E}_{13}|2\rangle \propto |2\rangle$ , which is only possible if  $a = 0$  and we have that

$$\hat{E}_{13} = |1\rangle\langle 3| - |3\rangle\langle 1| + |2\rangle\langle 2| \quad (21)$$

with eigenvectors

$$|\phi_{13}^0\rangle = |2\rangle \quad |\phi_{13}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|3\rangle \pm i|1\rangle). \quad (22)$$

With this choice we are led to

$$\hat{E}_{12}\hat{E}_{23} \neq \hat{E}_{13} \quad (23)$$

which clearly displays the quantum nature of this phase [27]. Note, in passing, that

$$[\hat{E}_{12}, \hat{R}_{23}] = [\hat{E}_{23}, \hat{R}_{12}] = 0 \quad (24)$$

and  $[\hat{R}_{23}, \hat{R}_{12}] = 0$ , so the interference between different channels (i.e., the noncommutativity of  $\hat{S}_{12}$  and  $\hat{S}_{23}$ ) is due to the noncommutativity of the corresponding phases.

#### 4. Positive operator measures for the qutrit phases

The unusual behaviour exhibited by the description of qutrit phases in terms of Hermitian operators can be considered to some extent exotic. One may think it preferable to represent qutrit phases by using a POVM taking continuous values in a  $2\pi$  interval.

We briefly recall that a POVM [28] associated with an observable  $\hat{\phi}$  is a set of linear operators  $\hat{\Delta}(\phi)$  ( $0 \leq \phi < 2\pi$ ), depending on the continuous parameter  $\phi$  and furnishing the correct probabilities in any measurement process through the fundamental postulate that

$$P(\phi) = \text{Tr}[\hat{\rho}\hat{\Delta}(\phi)]. \quad (25)$$

The real valuedness, positivity and normalization of  $P(\phi)$  impose

$$\hat{\Delta}^\dagger(\phi) = \hat{\Delta}(\phi) \quad \hat{\Delta}(\phi) \geq 0 \quad \int_0^{2\pi} d\phi \hat{\Delta}(\phi) = \hat{\mathbb{1}} \quad (26)$$

where the integral extends over any  $2\pi$  interval of the form  $(\phi_0, \phi_0 + 2\pi)$ ,  $\phi_0$  being a fiducial or reference phase. Note that, in general,  $\hat{\Delta}(\phi)$  are not orthogonal projectors as in the standard von Neumann measurements described by selfadjoint operators.

From our previous discussion, it is clear that we expect some complementarity between phases and inversions [29–31]. If we observe that

$$e^{i\phi' \hat{S}_{12}^z} = e^{-i\phi'/2} |1\rangle\langle 1| + e^{i\phi'/2} |2\rangle\langle 2| + |3\rangle\langle 3| \quad (27)$$

and argue that phase-shift operators must be  $2\pi$  periodic, we impose that any POVM  $\hat{\Delta}(\phi_{12}, \phi_{23})$  for a qutrit should satisfy

$$\begin{aligned} e^{i2\phi' \hat{S}_{12}^z} \hat{\Delta}(\phi_{12}, \phi_{23}) e^{-i2\phi' \hat{S}_{12}^z} &= \hat{\Delta}(\phi_{12} + \phi', \phi_{23}) \\ e^{i2\phi'' \hat{S}_{23}^z} \hat{\Delta}(\phi_{12}, \phi_{23}) e^{-i2\phi'' \hat{S}_{23}^z} &= \hat{\Delta}(\phi_{12}, \phi_{23} + \phi''). \end{aligned} \quad (28)$$

One can work out that the general POVM fulfilling these requirements must be of the form

$$\begin{aligned} \hat{\Delta}(\phi_{12}, \phi_{23}) &= \frac{1}{(2\pi)^2} \{ \hat{1} + [\gamma_{12} \exp(i(2\phi_{12} - \phi_{23})) |2\rangle\langle 1| + \gamma_{23} \exp(i(2\phi_{23} - \phi_{12})) |3\rangle\langle 2| \\ &\quad + \gamma_{13} \exp(i(\phi_{12} + \phi_{23})) |3\rangle\langle 1| + \text{h.c.}] \} \end{aligned} \quad (29)$$

where h.c. denotes Hermitian conjugate,  $\gamma_{ij} \leq 1$  are real numbers and  $\phi_{ij}$  is the relative phase between states  $|i\rangle$  and  $|j\rangle$ . These relative phases coincide precisely with the polar part of the realization of  $su(3)$  on the torus constructed in [27]. If we chose the  $\gamma_{ij}$  different, say  $\gamma_{12} = 1$  and the other two below the unity, then the expectation value of this POVM could reach the value zero for the superposition states  $(|1\rangle + \exp(i\theta)|2\rangle)/\sqrt{2}$ . However, for superpositions of states  $|1\rangle$  and  $|3\rangle$  or  $|2\rangle$  and  $|3\rangle$ , the expectation values of the POVM would always be greater than zero. Since there is no physical reason to assign special relevance to one specific superposition of the states, we assume that the POVM must be symmetric with respect to the states, which leads to

$$\gamma \equiv \gamma_{12} = \gamma_{23} = \gamma_{13}. \quad (30)$$

Moreover, we make henceforth the choice  $\gamma = 1$  because only for this choice can the POVM attain the expectation value zero for some particular state.

In contrast with the result of equation (23) formulated in terms of operators, now there are only two relevant phases in the qutrit description: the third can be inferred from the other two, as in the classical description.

The proposed POVM provides qutrit phases where any values of  $\phi_{12}$  and  $\phi_{23}$  are allowed. However, note that the probability density induced by this POVM can be written as

$$\begin{aligned} P(\phi_{12}, \phi_{23}) &= \frac{1}{(2\pi)^2} \{ 1 + [\rho_{12} \exp(i(2\phi_{12} - \phi_{23})) + \rho_{23} \exp(i(2\phi_{23} - \phi_{12})) \\ &\quad + \rho_{13} \exp(i(\phi_{12} + \phi_{23})) + \text{c.c.}] \} \end{aligned} \quad (31)$$

where  $\rho_{ij} = \langle i|\hat{\rho}|j\rangle$  and c.c. denotes complex conjugate. Therefore, this continuous range of variation is not effective in the sense that the values of  $P(\phi_{12}, \phi_{23})$  at every point  $(\phi_{12}, \phi_{23})$  cannot be independent, and we can find relations between them irrespective of the qutrit state. In other words, the complex parameters  $c_{ij}$  can be determined by the values of  $P(\phi_{12}, \phi_{23})$  at six points. Discreteness is inevitably at the heart of the qutrit phase [21].

Finally, we shall consider a remarkable example of POVM particularly suited to describe the qutrit phase. We recall that for a single-mode quantum field, a POVM for the field phase can be defined in terms of radial integration of quasiprobability distributions obtained using a coherent-state representation, much in the spirit of the classical conception [33, 34]. The natural generalization of this procedure to the qutrit problem involves the use of  $su(3)$  coherent

states. In the appendix we summarize the essential ingredients needed for this paper. Coherent states of a single qutrit are of the form

$$|\alpha, \beta\rangle = \frac{1}{\sqrt{C_{\alpha\beta}}}(|3\rangle + \alpha|2\rangle + \alpha\beta|1\rangle) \quad (32)$$

where  $\alpha$  and  $\beta$  are complex numbers and the normalization constant is

$$C_{\alpha\beta} = 1 + |\alpha|^2(1 + |\beta|^2). \quad (33)$$

These coherent states generate a POVM over the qutrit state space via the projectors  $|\alpha, \beta\rangle\langle\alpha, \beta|$ .

As shown in the appendix, the phases of  $\alpha$  and  $\beta$  are just those of  $\langle\alpha, \beta|\hat{S}_{32}|\alpha, \beta\rangle$  and  $\langle\alpha, \beta|\hat{S}_{21}|\alpha, \beta\rangle$ , respectively, while the phase associated with  $\langle\alpha, \beta|\hat{S}_{31}|\alpha, \beta\rangle$  is just the product of the other two. Let us write

$$\alpha = r_{23} e^{i\phi_{23}} \quad \beta = r_{12} e^{i\phi_{12}} \quad (34)$$

and integrate the projectors  $|\alpha, \beta\rangle\langle\alpha, \beta|$  radially over  $r_{12}$  and  $r_{23}$ , with respect to the measure (see equation (A.11))

$$d\mu = \frac{|\alpha|^2}{[1 + |\alpha|^2(1 + |\beta|^2)]^3} d^2\alpha d^2\beta. \quad (35)$$

After some calculations one obtains

$$\begin{aligned} \hat{\Delta}(\phi_{12}, \phi_{23}) = \frac{1}{(2\pi)^2} \left\{ \hat{1} + \frac{\pi}{96} [\exp(i(2\phi_{12} - \phi_{23}))|2\rangle\langle 1| + \exp(i(2\phi_{23} - \phi_{12}))|3\rangle\langle 2| \right. \\ \left. + \exp(i(\phi_{12} + \phi_{23}))|3\rangle\langle 1| + \text{h.c.}] \right\} \end{aligned} \quad (36)$$

which is just a specialized form of equation (29) and whose physical meaning is now clear.

## 5. Concluding remarks

In this paper, we have looked for possible descriptions of qutrit phases. Although it is possible to construct an extension of the Poincaré sphere to qutrits, the orthodox way of dealing with any observable is to represent them by selfadjoint operators. In this spirit, we have investigated a description of qutrit phases in terms of a proper polar decomposition of its amplitudes. Perhaps the most striking consequence of this description is that phases are discrete and do not commute.

We have also considered alternative generalized descriptions in terms of POVMs. In these descriptions, phases appear as parameters rather than operators. Additivity of phases follows from the commutativity of the Cartan elements. Although these POVMs reflect some desirable properties of the classical phase, they show an effective discreteness, even if in principle a continuous range of variation is assumed.

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### Appendix. $su(3)$ coherent states

In this appendix, we briefly summarize the essential ingredients of the construction of coherent states for three-level systems [35]. For concreteness, we shall consider fully symmetrical states of  $N$  three-level systems. In the Fock representation, we denote by  $|n_1, n_2, n_3\rangle$  the state in which there are  $n_1$  systems in level 1,  $n_2$  systems in level 2 and  $n_3$  systems in level 3. We observe that all these states can be generated from  $|0, 0, N\rangle$  by repeated application of the usual collective operators  $\hat{S}_{23}$  and  $\hat{S}_{12}$  (note that they coincide with (11), introduced for one qutrit, when  $N = 1$ ) as

$$(\hat{S}_{12})^n (\hat{S}_{23})^m |0, 0, N\rangle = \sqrt{\frac{N!m!}{(N-m)!}} \sqrt{\frac{m!n!}{(m-n)!}} |n, m-n, N-m\rangle \quad (\text{A.1})$$

with  $0 \leq n \leq m \leq N$ . Note that this is a simple extension of the relevant formula for the two-level case. In analogy with the atomic coherent states for  $su(2)$ , we define coherent states for qutrits as

$$|\alpha, \beta\rangle = \sqrt{\mathcal{N}_{\alpha\beta}} e^{\beta\hat{S}_{12}} e^{\alpha\hat{S}_{23}} |0, 0, N\rangle \quad (\text{A.2})$$

where  $\alpha$  and  $\beta$  are complex numbers and  $\mathcal{N}_{\alpha\beta}$  is a normalization constant that we shall write as

$$\mathcal{N}_{\alpha\beta} = \frac{1}{(\mathcal{C}_{\alpha\beta})^N} \quad (\text{A.3})$$

where we have introduced the real quantity

$$\mathcal{C}_{\alpha\beta} = 1 + |\alpha|^2(1 + |\beta|^2). \quad (\text{A.4})$$

In the Fock basis these states can be recast as

$$|\alpha, \beta\rangle = \sqrt{\mathcal{N}_{\alpha\beta}} \sum_{0 \leq n \leq m \leq N} \binom{N}{m}^{1/2} \binom{m}{n}^{1/2} \alpha^m \beta^n |n, m-n, N-m\rangle. \quad (\text{A.5})$$

After some calculations one gets the following mean values:

$$\begin{aligned} \bar{n}_3 &= \langle \alpha, \beta | \hat{S}_{33} | \alpha, \beta \rangle = \frac{N}{\mathcal{C}_{\alpha\beta}} \\ \bar{n}_2 &= \langle \alpha, \beta | \hat{S}_{22} | \alpha, \beta \rangle = \frac{N}{\mathcal{C}_{\alpha\beta}} |\alpha|^2 \\ \bar{n}_1 &= \langle \alpha, \beta | \hat{S}_{11} | \alpha, \beta \rangle = \frac{N}{\mathcal{C}_{\alpha\beta}} |\alpha|^2 |\beta|^2 \end{aligned} \quad (\text{A.6})$$

which immediately shows that the ratios of the average population numbers are given by

$$\bar{n}_3 : \bar{n}_2 : \bar{n}_1 = 1 : |\alpha|^2 : |\alpha|^2 |\beta|^2. \quad (\text{A.7})$$

On the other hand, one can also compute

$$\begin{aligned} \langle \alpha, \beta | \hat{S}_{32} | \alpha, \beta \rangle &= \frac{N}{\mathcal{C}_{\alpha\beta}} \alpha \\ \langle \alpha, \beta | \hat{S}_{21} | \alpha, \beta \rangle &= \frac{N}{\mathcal{C}_{\alpha\beta}} |\alpha|^2 \beta \\ \langle \alpha, \beta | \hat{S}_{31} | \alpha, \beta \rangle &= \frac{N}{\mathcal{C}_{\alpha\beta}} \alpha \beta. \end{aligned} \quad (\text{A.8})$$

The phases of  $\alpha$  and  $\beta$  are then just those of  $\langle \alpha, \beta | \hat{S}_{32} | \alpha, \beta \rangle$  and  $\langle \alpha, \beta | \hat{S}_{21} | \alpha, \beta \rangle$ , respectively. Note, in passing, that the third phase associated with  $\langle \alpha, \beta | \hat{S}_{31} | \alpha, \beta \rangle$  is just the product of the other two, as it happens in classical physics.

The atomic coherent states with different amplitudes are not orthogonal

$$\langle \alpha_1, \beta_1 | \alpha_2, \beta_2 \rangle = \frac{[1 + \alpha_1^* \alpha_2 (1 + \beta_1^* \beta_2)]^N}{[1 + |\alpha_1|^2 (1 + |\beta_1|^2)]^{N/2} [1 + |\alpha_2|^2 (1 + |\beta_2|^2)]^{N/2}} \quad (\text{A.9})$$

but form an overcomplete set. In fact, it is easy to verify the following resolution of the identity:

$$\frac{(N+1)(N+2)}{\pi^2} \int d\mu |\alpha, \beta\rangle \langle \alpha, \beta| = \hat{1} \quad (\text{A.10})$$

where the measure  $d\mu$  is

$$d\mu = \frac{|\alpha|^2}{[1 + |\alpha|^2 (1 + |\beta|^2)]^3} d^2\alpha d^2\beta. \quad (\text{A.11})$$

The above discussion pertains only to the fully symmetric subspace. Of course, it is enough for our purposes, although it can be extended to other subspaces in a very direct way.

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